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The eigenvalue approach to solve the groundwater flow partial differential equation: Some new analytical solutions

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Introduction

For systems with a linear behavior (i.e. transmissivity, storativity and boundary conditions invariant on time and independent of the saturated depth), the eigenvalue method produces an efficient solution to the groundwater flow partial differential equation, which has found important applications. Examples of such applications are (i) the realistic inclusion of aquifers in conjunctive-use models of complex systems, beyond the single-cell aquifer representation generally used in this type of models [1], (ii) determining a discrete in space, continuous in time, solution to the groundwater flow equation in real cases [1,3], (iii) preliminary evaluation, through analytical solutions, of the influence of pumping in stream depletion for simple-to-moderately complex geometries when aquifer data is scarce [2,4].

This work presents additional analytical solutions for finite aquifers. The solution is expressed as a linear combination of orthogonal components, what allows interpreting each aquifer as composed by a series of linear independent cells.

Eigenvalue solution to the groundwater flow equation

In 2-D, the groundwater flow partial differential equation is

$$\mathcal{L}(h) + Q(x,y) = S(x,y)\frac{\partial h}{\partial t}, \tag{1}$$

plus adequate initial and boundary conditions, with $\mathcal{L}(\cdot)$ is a lineal operator

$$\mathcal{L}(\cdot) = \frac{\partial}{\partial x}\left(T_x(x,y)\frac{\partial \cdot}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y(x,y)\frac{\partial \cdot}{\partial y}\right),$$

$h(x,y,t)$ is the piezometric head $[L]$, $T_x(x,y)$ and $T_y(x,y)$ are the diagonal components of the transmissivity tensor (the principal components of which are aligned with the Cartesian axes of co-ordinates) $[L^2T^{-1}]$, $S(x,y)$ is the storage coefficient $[-]$, and $Q(x,y)$ are the external stresses $[LT^{-1}]$, which may include areal recharge/withdrawal as well as punctual stresses.

The solution of equation (1) is given by Sahuquillo [3] as

$$h(x,y,t) = \sum_{i=0}^{\infty} l_i(t)A_i(x,y),$$

with

$$l_i(t) = \frac{1-e^{-\alpha_i t}}{\alpha_i}\int_\Omega Q(x,y)A_i(x,y)d\Omega,$$

with Ω representing the aquifer domain. Eigenfunctions A_i $[L^{-1}]$, and eigenvalues α_i $[T^{-1}]$ are the basic components of the solution of a classical Sturm-Liouville problem

L(A$_i(x,y)) + \alpha_i S(x,y)A_i(x,y) = 0$,

into which the groundwater flow partial differential equation can be rewritten after some manipulation [3].

Eigenfunctions $A_i(x,y)$ conform an orthogonal base with respect to the storage coefficient.

$$\int_\Omega A_i(x,y)S(x,y)A_j(x,y)d\Omega = \delta_{i,j},$$

with $\delta_{i,j}$ being the Kronecker delta.

For a given t, values $l_i(t)$ $[L^2]$ can be interpreted as the state representation of the aquifer in the eigenfunction base. In this respect, the aquifer can be considered as made of an infinite number of cells, each cell characterized by its $l_i(t)$ function. This decomposition of the solution of the groundwater flow equation as an infinite sum of independent components yields a very interesting interpretation of the aquifer behavior:

- The total volume of the aquifer at any given time t is distributed between the infinite number of cells according to the following expression

$$V_i(t) = l_i(t)F_i,$$

  with

$$F_i = \int_\Omega S(x,y)A_i(x,y)d\Omega.$$

- If a river is connected to the aquifer, the inflow/outflow is also distributed between the cells according to

$$Q_{r_i}(t) = \alpha_i V_i(t),$$

  with Q_{r_i} the inflow/outflow to the river associated to cell i.

- The total stress is also distributed between the cells as follows; if Q is the total stress, the fraction that applies to cell i is $b_i Q$ with

$$\begin{aligned} Q &= \int_\Omega Q(x,y)d\Omega, \\ b_i &= \frac{\int_\Omega Q(x,y)A_i(x,y)d\Omega}{Q}F_i. \end{aligned}$$

It is easily demonstrated that $\sum_{\text{ß}=0}^{\infty} b_i = 1$.

Each of these cells, at any given time, has an associated volume $V_i(t)$ given by

$$V_i(t) = l_i(t)F_i,$$

with

$$F_i = \int_\Omega S(x,y)A_i(x,y)d\Omega.$$

After some manipulation, for an aquifer with an initial volume V_0, the expressions for the distribution of the inflow/outflow to the river and for the volume of each cell is given by

$$\begin{aligned} Q_{r_i}(t) &= b_i Q(1-e^{-\alpha_i t}) \\ V_i(t) &= V_{i_0}e^{-\alpha_i t} + b_i Q\frac{1-e^{-\alpha_i t}}{\alpha_i} \end{aligned}$$

with

$$V_{i_0} = F_i \int_\Omega h_0(x,y)S(x,y)A(x,y)d\Omega$$

vectors $\{l_i(t), V_i(t), Q_{r_i}(t), i = 0, \ldots, \infty\}$ can be regarded as vector states of the aquifer fully defining its state, as $h(x,y,t)$ does. The great interest of this representation is that, in most applications of aquifer-river interaction, only a few components are necessary to yield a good approximation of the aquifer state continuously in time. (This is particularly true when wells are not too close to the river.)

The above discussion refers to the analyticial solution of the grounwater flow equation. But, for those cases in which the Sturm-Liouville problem cannot be solved analytically, a numerical solution can be obtained by standard finite difference or finite element solutions. In such cases, the number of eigenvalues equals the number of discretizing cells or nodes; and instead of eigenfunctions, each eigenvalue has an eigenvector associated to it. The interpretation of the aquifer as an ensemble of cells remains valid.

Conclusions

The eigenvalue approach provides a generic solution for the groundwater flow in any aquifer with a linear behavior. This solutions is amenable to a physical interpretation that helps in the understanding of the aquifer evolution and its relation with rivers. Besides, it can be used to obtain analytical solutions for finite aquifers in cases for which only 1D solutions existed. In this work, analytical solutions for the eigenvalues and eigenfunctions for new cases are provided:

- Rectangular homogeneous aquifer perfectly or partially connected to one, two or three rivers. In the case of two rivers, they can be parallel or intersecting.
- Rectangular aquifers with two bands of different properties.
- Circular sector aquifers.
- Rectangular aquifers with a partially-connected river running parallel to two sides, not necessarily through the center line, and with a finite (larger than zero) river width.

References

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