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Analytical solutions of partial differential equations in a homogeneous and isotropic two-dimensional aquifer

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Introduction

According to hydrology, water flowing in an aquifer follows the three-dimensional partial differential parabolic equation (1) [see e.g. D. K. Todd: Groundwater Hydrology, Wiley and Sons, p.100 and p. 123 as quoted later]:

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \cdot \frac{\partial^2 h}{\partial y^2} + K_z \cdot \frac{\partial^2 h}{\partial z^2} = S \cdot \frac{\partial h}{\partial t} \quad (1)$$

where: h is the piezometric head; K_x, K_y, K_z are the hydraulic conductivities; S is the specific storage.

For a vertical well in an aquifer, often isotropy can be assumed, so that all conductivities are equal to a unique value K. Under such assumption Darcy's Law is:

$$\underline{J} = -K \cdot \underline{\nabla} h \quad (2)$$

where $\underline{J}$is the vector flux. If moreover h is assumed to be mainly linearly dependent on height, equation (1) becomes the following two-dimensional parabolic equation on piezometric head [assumed to be evaluated at height z=0 along vertical axis and to depend on radial distance r (from vertical axis) and on time t], [p. 123 and foll.]:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial h}{\partial r} = \frac{S}{K} \cdot \frac{\partial h}{\partial t} = \frac{1}{\chi} \cdot \frac{\partial h}{\partial t} \quad (3)$$

Digital fields computed by equations (1), (3) often appear unstable due to the parabolic nature of equation. Therefore it may be useful to compare digital fields to more precise analytical solutions.

Table of analytical two-dimensional solutions

Solutions are obtained by assuming a product of two functions: one depending only on time and the other one depending only on the following variable u containing time and distance r:

$$u = \frac{r^2}{4 \cdot \chi \cdot t} \quad (4)$$

By introducing such factorisation into the parabolic equation (3), an equality can be obtained. The left hand side (l.h.s.) of equality contains a first order ordinary differential equation on a function of time; whereas the right hand side (r.h.s.) of equality contains a second order ordinary differential equation on a function of the new variable u. Therefore l.h.s. and r.h.s. must be equal to a same separation constant. By solving first order ordinary differential equation (for a particular separation constant), only one function of time is obtained; whereas by solving second order ordinary differential equation (for the same separation constant) two functions of u are obtained. Each function of u involves unconventional functions as: E functions; Laguerre polynomials (alternate sequence of monomials having positive and negative signs); modified Laguerre polynomials (same of previous polynomials but all monomials have positive signs); Ein function; a function G appropriately defined for parabolic equation (3).

Such analytical solutions provide also water flowing from well at the appropriate height in aquifer. In this abstract a Table is shown which provides some simple solutions and also General Formulas. The list of solutions contains: the function of time; the first and second function of u; the corresponding value of the separation constant (Const.) [-j,...;-3;-2;-1;0;+1;+2;+3;..., +j]. Theis's solution quoted in p. 123, corresponds to the second solution for separation constant equal 0.

Moreover each solution in the list needs an appropriate normalization constant.

Table

$Const.$	First solution	Second solution
-j	$\frac{1}{(4\cdot\chi\cdot t)^j}\cdot e^{-u}\cdot\left[\frac{1}{j!}\cdot L_{j-1}(u)\right]$	$\frac{1}{(4\cdot\chi\cdot t)^j}\cdot\sum_{k=0}^{j-1}\begin{pmatrix} j-1 \\ k \end{pmatrix}\cdot u^k\cdot\frac{d^k(G)}{du^k}$
......	......	......
-3	$\frac{1}{(4\cdot\chi\cdot t)^3}\cdot e^{-u}\cdot\left[1-2u+u^2\right]$	$\frac{1}{(4\cdot\chi\cdot t)^3}\cdot\left[G+2uG^{'+u^2G^{''}}\right]$
-2	$\frac{1}{(4\cdot\chi\cdot t)^2}\cdot e^{-u}\cdot[1-u]$	$\frac{1}{(4\cdot\chi\cdot t)^2}\cdot\left[G+uG^{'}\right]$
-1	$\frac{1}{(4\cdot\chi\cdot t)}\cdot e^{-u}$	$\frac{1}{(4\cdot\chi\cdot t)}\cdot e^{-u}\cdot[Ein(-u)-\log(u)]=\frac{1}{(4\cdot\chi\cdot t)}\cdot G(u)$
0	1	$E_1(u)$
+1	$(4\cdot\chi\cdot t)\cdot[1+u]$	$(4\cdot\chi\cdot t)\cdot[E_1(u)-E_2(u)]$
+2	$(4\cdot\chi\cdot t)^2\cdot\left[1+2u+\frac{1}{2}u^2\right]$	$(4\cdot\chi\cdot t)^2\cdot[E_1(u)-2E_2(u)+E_3(u)]$
+3	$(4\cdot\chi\cdot t)^3\cdot\left[1+3u+\frac{3}{2}u^2+\frac{1}{6}u^3\right]$	$(4\cdot\chi\cdot t)^3\cdot[E_1(u)-3E_2(u)+3E_3(u)-E_4(u)]$
......	......	......

In the table:

$E_n(u)=u^{n-1}\int\limits_u^\infty\frac{e^{-u^{'}}}{u^{'n}}du^{'}$	n=1,2,3,.... e.g.: $E_1=\int\limits_u^\infty\frac{e^{-u^{'}}}{u^{'}}du^{'}$

$Ein(u)=\int\limits_0^u\frac{1-e^{-u^{'}}}{u^{'}}du^{'}$	$G(u)=e^{-u}\left[Ein\left(-u\right)-\log\left(u\right)\right]$

$L_{j-1}(u)$ = Laguerre Polynomial order (j-1): $\left[L_{j-1}(u)=e^u\cdot\frac{d^{(j-1)}}{du^{(j-1)}}\left(u^{-(j-1)}\cdot e^{-u}\right)\right]$

$L_j^*(u)$ =modified Laguerre Polynomial order (j):$\left[L_j^*(u)=e^{-u}\cdot\frac{d^j}{du^j}\left(u^j\cdot e^u\right)\right]$

The following quantity is introduced,

which represents the density of water per unit length of well, per unit time, which could be withdrawn from aquifer or injected into aquifer from the well. Such density vanishes for all First solutions shown in the Table. Therefore such solutions are not

$$\boxed{\left\{2\pi r\left[-K\left(\frac{\partial h}{\partial r}\right)\right]\right\}_{r\to 0}}$$

suited to describe a real well. Instead such density is proportional to $(4 \cdot \chi \cdot t)^j$ for each Second solution (negative j, 0, positive j). Therefore Second solutions are suited for description of a well.