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A Multi-Object Singularity Method for Potential Problems in heterogeneous Continuum Media

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1 Introduction

Boundary value potential problems for the Laplace or Poisson equation in continuum media appear as mathematical formulation of processes in different areas of physics, e. g. in electrostatics and electrodynamics, heat conduction and fluid flows through porous media. In this paper the general case of a bounded continuum media constituted of several arbitrary distributed sub-domains with different conductivities (e.g. thermal conductivity or hydraulic conductivity) and several physical source or sink sub-domains which induce flow, transfer or transport processes (e.g. heat sources in the case of heat transfer, wells or drains in the case of groundwater flow) is considered. For modeling such processes a meshless Multi-Object Singularity Method (MOSM) is proposed. The objects are the sub-domains with different conductivities called Non-Singularity-Objects (NSO) and the source/sink sub-domains are called Singularity-Objects (SO). The SO can be located arbitrarily regarding the NSO. On the boundary of the NSO transition conditions between the internal and external transport/flow should be satisfied while on the boundary of the SO only boundary conditions for the external process are required. The singular integral representation of the solution of boundary value problems for each object will be used as theoretical base of the MOSM.

It is well understood how to formulate boundary integral equations for those potential problems. However, most publications on this subject lack suggestions on how to handle the arising singular integral kernels with respect to generated acceptable results in the actual implementations. In this paper we present a method (i.e. MOSM) including software implementation aspects for modeling 3D flow, transfer or transport processes in heterogeneous continuum media and its reliability in practical issues. The presented MOSM will be focused especially on the flow problems in porous media.

2 Formulation of the boundary value problem and its solution by means of singular integral representations

One considers a bounded continuum domain Ω with the boundaryΓ. The domain Ω includes several Non-Singularity-Objects (NSO) and Singularity-Objects (SO) as well as defined sub-domains:

$$\Omega^+_{iNSO} \subset \Omega, \;\; i = 1, 2, ..., n_{NSO}$$

with the closed boundaries Γ_{iNSO} and

$$\Omega^+_{jSO} \subset \Omega, \;\; j = 1, 2, ..., n_{SO}$$

with the closed boundaries Γ_{jSO}.

In terms of groundwater flow in porous media the NSO represent inclusions with different hydraulic conductivities which can be also impervious or a hole filled with water. SO represent discharge or recharge objects like wells, drains or ditches located arbitrarily and shaped like a stretched tube.

The flow domains are the interiors of all NSO (i.e. Ω^+_{NSO}) and their complementary region to Ω (i.e. Ω^+) are defined as follow:

$$\Omega^+_{NSO} = \bigcup_{i=1}^{n_{NSO}} \Omega^+_{iNSO}$$

and

$$\Omega^+ = \Omega - \Omega^+_{NSO} \cup (\bigcup_{j=1}^{n_{SO}} \Omega^+_{jSO}) \tag{1}$$

The searched solution is the potential function

$$\varphi(x) = \begin{cases} \varphi^+_{iNSO}(x), & x \in \Omega^+_{iNSO}, \;\; i = 1, 2, ..., n_{NSO} \\ \varphi^+(x), & x \in \Omega^+ \end{cases} \tag{2}$$

with ϕ(x) satisfying the Laplace equation in each field point x

$$\nabla^2 \varphi(x) = 0, \;\; x \in \Omega^+_{NSO} \cup \Omega^+ \tag{3}$$

and the following boundary and contact conditions on the boundaries

$$\varphi^+_{|\Gamma_\varphi} = \varphi_0$$

on the potential boundary Γ_φ of the domain Ω,

$$\frac{\partial \varphi^+}{\partial n}_{|\Gamma_q} = q_B$$

on the flux boundary Γ_q of the domain Ω;

$$\varphi^+_{|\Gamma_{jSO}} = \varphi_{0jSO}$$

on the SO-boundaries,

$$\varphi^+_{|\Gamma_{iNSO}} = \varphi^+_{iNSO|\Gamma_{iNSO}}$$

and

$$\frac{\partial \varphi^+}{\partial n}_{|\Gamma_{iNSO}} = \frac{\partial \varphi^+_{iNSO}}{\partial n}_{|\Gamma_{iNSO}} \tag{4}$$

on the NSO boundaries where

$$\Gamma = \Gamma_\varphi \cup \Gamma_q.$$

$\varphi_0, \quad q_B$ and φ_{0jSO} are given functions.

3 and 4 build a boundary value problem for the searched potential function ϕ.

Mesh dependent methods, namely the Finite Differences Method (FDM) and the Finite Element Method (FEM) are able to obtain approximate descriptions of the domain geometry and the governing equations. Their major disadvantage is their tendency to generate extensively huge data sets and equation systems for three-dimensional problems. Another disadvantage appears when we put these mesh dependent methods into practice for NSO and SO. It is difficult if not completely impracticable to describe arbitrary distributed NSO and SO using FEM/FDM because it requires an update or costly regeneration of the 3D mesh after each modification of the external boundary or of the interior objects (i.e. the distribution and the shape of NSO and SO).

A more efficient approach to solve such complex potential problems is to find an adequate method which allows the determination of the required internal/external potential functions. It is possible by means of boundary integral representations using only boundary elements or distributed singularities to shape NSO and SO respectively.

The well known fundamental solution of the 3D potential problem i.e. the Newtonian potential generated at a field point x from a unit simple source located at a source point ξ:

$$\omega = \frac{1}{4\pi(\xi, x)}$$

with r(ξ, x) as the distance between ξ and x (see [BANERJEE 1994] and [BREBBIA, TELLES, WROBEL 1984]) is applied. The searched functions $\varphi^+_{iNSO}(x)$ and $\varphi^+(x)$ can be represented as surface single layer potentials like in the case of 2D problems analyzed by [DAVID 1995].

$$\varphi^+_{iNSO}(x) = \frac{1}{4\pi} \int\limits_{\Gamma_{iNSO}} \frac{\psi_{\Gamma_{+iNSO}}}{r(\xi, x)} d\Gamma + c_i, \quad x \in \Omega^+_{iNSO}$$

and

$$\varphi^+(x) = \frac{1}{4\pi} \int\limits_{\Gamma} \frac{\psi_\Gamma}{r(\xi, x)} d\Gamma + \frac{1}{4\pi} \sum_i \int\limits_{\Gamma_{iNSO}} \frac{\psi_{\Gamma_{-iNSO}}}{r(\xi, x)} d\Gamma + \sum_j \int\limits_{l_j^{(n)}} \frac{\psi_j}{r(\xi, x)} dl^{(n)} + c \tag{5}$$

In these integral representations $\psi^+_{+iNSO}(\xi)$, $\psi^+_{-iNSO}(\xi)$, $\psi^+_\Gamma(\xi)$ and $\psi^+_j(\xi)$ are unknown density distributions along the different boundaries. In the same representation $l_j^{(n)}$ are the spatial supports of the singularities to generate Singularity Objects (SO): point singularities (n=0), line singularities (n=1) or surface singularities (n=2).

Using the integral representations 5 and taking into account the boundary conditions 4 one obtains a set of integral equation which allows the determination of the density functions $\psi^+_{+iNSO}(\xi)$, $\psi^+_{-iNSO}(\xi)$, $\psi^+_\Gamma(\xi)$ and $\psi^+_j(\xi)$, and furthermore the determination of the searched potential functions 2.

3 Numerical solution method and implementation

Because only very simple geometrical models can be handled analytically, we have to solve the multiple-boundary value problem by discretisation. So it is necessary to introduce boundary elements which forms domain boundaries and for the case of SO, (n-dimensional) sink/source analytical elements. For practical reasons integrations are calculated numerically for each element, except when self-influencing.

The transition conditions have to be satisfied on Γ_{NSO}, on Γ_{SO} and Γ, which implies the solution of singular integral equation (of Fredholm first kind kernels). If the field point x on which φ or $\frac{\partial\varphi}{\partial n}$ have to be determined lies on an element surface s with $x \in s$, the integration has to be done numerically by special quadrature schemes (see [BREBBIA, TELLES, WROBEL 1984]) or analytically in the sense of Cauchy principal values (see [GAKHOV 1990]).

If the point of observation $x \in \Omega^+ \cup \Omega^+_{NSO}$ gets close to a boundary Γ or Γ_{NSO} but with $x \notin \Gamma \cup \Gamma_{NSO}$, usually regular quadrature schemes are used. This results in error distributions close to the boundary. If x gets closer to a boundary point, φ rises up to infinity. This effect, of course, is not a physical problem but a problem of discretisation and numerical integration.

To smoothen the potential distribution in these areas we propose the introduction of an additional set of contours

$$\Gamma^{+\varepsilon} \supset \Gamma$$

for the outer boundary,

$$\Gamma^{+\varepsilon}_{iNSO} \supset \Gamma_{iNSO}$$

and

$$\Gamma^{-\varepsilon}_{iNSO} \subset \Gamma_{iNSO} \tag{6}$$

for NSO,

where the index $\pm\varepsilon$ denotes a distance in direction of the outward normal vector of the referenced domain. (For the separation of source distributions and boundary contours see e.g. [BISCHOFF 1977] and [POZRIKIDIS 1992].) The unknown density distribution ψ is now calculated for the indexed boundaries and the SO sources in a way that it satisfies the boundary and transition conditions on all unindexed boundaries:

$$\varphi^+_{iNSO}(x) = \frac{1}{4\pi} \int\limits_{\Gamma^{+\varepsilon}_{iNSO}} \frac{\psi_{\Gamma^{+\varepsilon}_{iNSO}}}{r(\xi,x)} d\Gamma + c_i, \;\; x \in \Omega^+_{iNSO} \tag{7}$$

$$\varphi^+(x) = \frac{1}{4\pi} \int\limits_{\Gamma^{+\varepsilon}} \frac{\psi_{\Gamma^{+\varepsilon}}}{r(\xi,x)} d\Gamma + \frac{1}{4\pi} \sum_i \int\limits_{\Gamma^{-\varepsilon}_{iNSO}} \frac{\psi_{\Gamma^{-\varepsilon}_{iNSO}}}{r(\xi,x)} d\Gamma + \sum_j \int\limits_{l^{(n)}_j} \frac{\psi_j}{r(\xi,x)} dl^{(n)} + c \tag{8}$$

Depending on the magnitude of the distance ε and the distribution of quadrature points we are able to obtain quite smooth potential distributions throughout Ω^+ and Ω^+_{NSO}, without requiring special treatment of singular Fredholm kernels (due to $r(\xi,x) \neq 0$).

4 Conclusion

Having proposed a way of domain composition by the introduction of Singularity-Objects (SO) and Non-Singularity-Objects (NSO) for the use of a Multi-Object Singularity Method (MOSM) we are able to solve potential problems of arbitrary geometrical configuration of 3D heterogeneities elegantly. We solve singular integral by distributing the density contours on additional boundaries, which lies outside or inside the real boundaries. As an extra advantage, this generates smooth potential distribution even close to and on the boundaries after discretisation and numerical integration. This meshless method allows the reduction of the discretization complexity.

5 References

[BANERJEE 1994] P. K. Banerjee: "The Boundary Element Methods in Engineering", McGraw-Hill 1994.

[BISCHOFF 1977] H. Bischoff: "Die Berechnung von Potentialfeldern mit der Randintegralmethode dargestellt am Beispiel der ebenen stationären Grundwasserbewegung", Technischer Bericht Nr. 18 aus dem Institut für Hydraulik und Hydrologie der Technischen Hochschule Darmstadt 1977.

[BREBBIA, TELLES, WROBEL 1984], C. A. Brebbia, J. C. F. Telles, L. C. Wrobel: "Boundary Element Techniques", Springer 1984.

[DAVID, GERDES 1995] I. David, H. Gerdes: Incorporation of local three-dimensional flow in plane BEM, BEM XVII-USA, Computational Mechanics Publications, pg.449-458, Southamton, Boston 1995.

[DAVID, GERDES 1998] I. David, H. Gerdes: Coupling Analytical Element, BEM and FEM to develop a model for groundwater flow. Computational Mechanics Publications, Computational Methods in Water Resources 1998, vol. 1, pg. 362-370.

[DAVID 2004] I. David: Analytical Element Method for Modeling coupled ground water flow generated by drains and partially penetrating wells, 4^{th} International Conference on the Analytic Element Method, November 2003, Saint-Etienne.

[GAKHOV 1990] F. D. Gakhov: "Boundary value problems", Dover Publications 1990.

[HARTMANN 1989] F. Hartmann: "Introduction to Boundary Elements", Springer 1989.

[POZRIKIDIS 1992] C. Pozrikidis: “Boundary integral and singularity methods for linearized viscous flow”, Cambridge University Press 1992.