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Self-Similarity of PDF and its Violation in the Theory of fully developed Turbulence

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The Kolmogorov equation

$$\left\langle U^3 \right\rangle = -\left(4/5\right)\overline{\varepsilon}r + 6\nu\left(d\left\langle U^2\right\rangle/dr\right) \tag{1}$$

relates the second-order longitudinal structure function $D_{ll}\left(r\right) = \left\langle U^2\right\rangle$ and the third-order longitudinal structure function $D_{lll}\left(r\right) = \left\langle U^3\right\rangle$ [1]. Here, $U\left(r\right)$ is the difference of longitudinal components of velocity at the distance r in locally homogeneous, isotropic, and stationary turbulence, $\overline{\varepsilon}$ is the mean value of the energy dissipation per unit of mass, and ν is the kinematic viscosity. The equation (1) relates two different unknown functions, $D_{ll}\left(r\right)$ and $D_{lll}\left(r\right)$, i.e., it is unclosed.

Nevertheless, it is possible to present (1) as the closed equation, if we express the unknown quantities $\left\langle U^n\left(r\right)\right\rangle$ in terms of the corresponding probability density function. Let us introduce the probability density function (PDF) for U, which also depends on separation r: $W\left(U,r\right)$. In terms of W we have

$$\left\langle U^2\left(r\right)\right\rangle = \int\limits_{-\infty}^{\infty} U^2 W\left(U,r\right)dU, \quad \left\langle U^3\left(r\right)\right\rangle = \int\limits_{-\infty}^{\infty} U^3 W\left(U,r\right)dU \tag{2}$$

and (1) takes the form

$$\int\limits_{-\infty}^{\infty}\left[6\nu U^2\left(\partial W\left(U,r\right)/\partial r\right) - U^3 W\left(U,r\right)\right]dU = \left(4/5\right)\overline{\varepsilon}r. \tag{3}$$

The integro-differential equation (3) contains only one unknown function $W\left(U,r\right)$ (i.e., it is closed), but this equation does not uniquely determine $W\left(U,r\right)$. On the other hand, the unknown precise $W\left(U,r\right)$ must satisfy the equation (3) .

In general, the function $W(U,r)$ may depend on the separation r in two different ways:

1. Due to the dependence of the characteristic scale of W (with respect to U) on the rms value $\sqrt{D_{ll}(r)}$. We refer to this part as self-similar dependence.

2. Due to the changing shape of $W(U,r)$ while r changes (not a self-similar part).

Thus, in general,

$$W(U,r) = [D_{ll}(r)]^{-1/2} F\left(U/\sqrt{D_{ll}(r)}, r\right). \tag{4}$$

If we neglect the dependence of F on the second argument r, i.e., if we assume the self-similarity of the PDF, $W(U,r) = [D_{ll}(r)]^{-1/2} F\left(U/\sqrt{D_{ll}(r)}\right)$, the Kolmogorov equation (3) reduces to the nonlinear differential equation

$$6\nu\left(dD_{ll}(r)/dr\right) + S\,[D_{ll}(r)]^{3/2} = (4/5)\,\bar{\varepsilon}r, \quad D_{ll}(0) = 0 \tag{5}$$

with respect to $D_{ll}(r)$. Here, $-S$ is the skewness of $W(U,r)$, which is independent of distance r for self-similar PDF. The equation (5) was considered by Obukhov [2]. It follows from (5) that asymptotically for $r \to \infty$,

$$D_{ll}(r) \to (45S)^{2/3}\,(\bar{\varepsilon}r)^{2/3}, \tag{6}$$

i.e., the Kolmogorov-Obukhov 2/3 law follows from the Kolmogorov equation (1) without using the dimension analysis described in [3]. Numerical solution of the equation (5) shows that the spectrum $f(k)$, which is related to $D_{ll}(r)$ by the formula

$$D_{ll}(r) = 2\int_{-\infty}^{\infty} [1 - \cos(kr)]\, f(k)\, dk \tag{7}$$

is positive. The compensated spectrum $k^{5/3} f(k)$ has a maximum in the transition range between the inertial and viscous sub-ranges (the known bottle-neck effect).

Any possible deviations from the 2/3 law in fully-developed turbulence must be related to the dependence of $F\left(U/\sqrt{D_{ll}(r)}, r\right)$ on the second argument, r, i.e., with violation of self-similarity of PDF. Experimental data show that such dependence really takes place in the region of large values of $|U| > 4\sqrt{D_{ll}(r)}$ and may be very important for high-order structure functions and for intermittency effect. We consider a simple two-component model

$$W(U,r) = [D_{ll}(r)]^{-1/2}\left\{p(r)\,F_1\left(U/\sqrt{D_{ll}(r)}\right) + [1 - p(r)]\,F_1\left(U/\sqrt{D_{ll}(r)}\right)\right\}, \tag{8}$$

where $p(r)$ and $1 - p(r)$ are the probabilities of realizing distributions F_1 and F_2, respectively. Such a model may correspond to fluctuations of values of ε, averaged over distance r.

It is possible to choose such function $p(r)$, $0 \leqslant p(r) \leqslant 1$ that the asymptotic behavior of $D_{ll}(r)$ for $r \to \infty$ has the form

$$D_{ll}(r) \to Constant \times r^{2/3+\alpha}, \tag{9}$$

which corresponds to intermittent turbulence. It is shown that this very simple two-component model may describe the intermittency effect and allows us to obtain good estimations of the exponents $\kappa(\rho)$ in the functions $\langle |U(r)|^{\rho} \rangle \sim r^{\kappa(\rho)}$. In the following table we present the local exponents κ for different ρ and the corresponding experimental data obtained by A. Praskovsky (unpublished).

Table 1. Local exponents at the separation $r = 10^4 \nu^{3/4} \overline{\varepsilon}^{-1/4}$

ρ	0.2	0.6	1.0	1.4	1.8	2.0	3.0	4.0
κ	.071	.213	.355	.497	.639	.710	1.063	1.414
Experiment	.081	.237	.386	.530	.668	.736	1.053	1.341
Error, $\pm\sigma_\kappa$	.001	.004	.009	.013	.017	.019	.025	.033

The spectrum $f(k)$ obtained for this model also describes the bottle-neck effect, which agrees with the experimental data of [4].

It is important that the same model of PDF in combination with the 4/5 Kolmogorov equation allows us to describe both the exponents in the inertial range and the bottle-neck effect in the viscous range.

The paper [5] contains all related details.

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