Bayesian Parameter Inference in Instantaneous Mantle Flow with Plates

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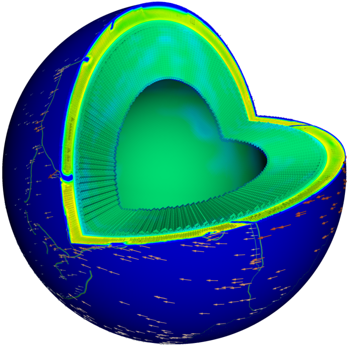
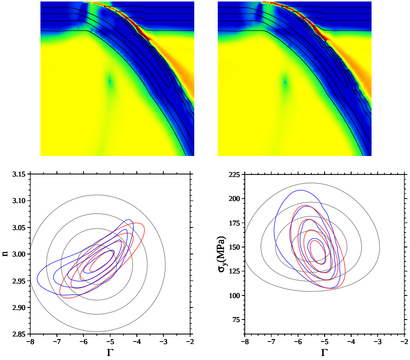
Goal: Better understanding of fundamental mechanisms

Ⓘ Instantaneous mantle flow & associated plate tectonics with realistic parameters & resolutions to faulted plate boundaries (“the forward problem”).

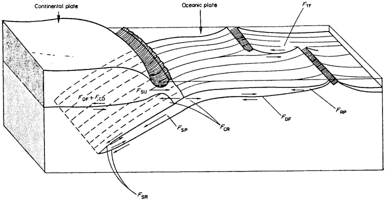


ⓘⓘ Parameter/model inference from observations+mantle flow models.



What drives plate motions?

- How does slab pull really work?
- What's the role of back arc-spreading in global mantle motions?
- Can one infer the degree of mechanical plate coupling between subduction zones based on plate motions?
- What are the trade-offs of underlying forces [Conrad/Hager '99 using scaling arguments]:
    - buoyancy from slabs
    - resistance by inter-plate faults
    - drag by the underlying mantle
    - resistance by bending the oceanic lithosphere



Approach

- Instantaneous high-resolution model, one time instant (present day, or historical)

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Target is to infer the most uncertain parameters (rheological parameters, plate coupling strength) to global observations (Bayesian inverse problem).

High-resolution global forward models

Equations for momentum/mass conservation

$$-\nabla \cdot \Big[\mu(T, \boldsymbol{u})\,(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)\Big] + \nabla p = \boldsymbol{f}(T)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$\boldsymbol{u}$ ... velocity

p ... pressure

T ... temperature

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Rheology is shear-thinning with plastic yielding, and upper/lower viscosity bounds; exponential dependence on temperature:

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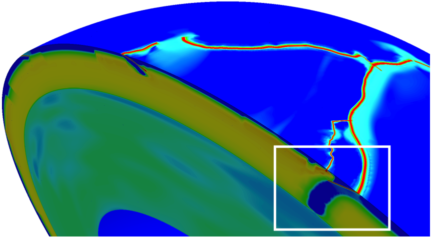
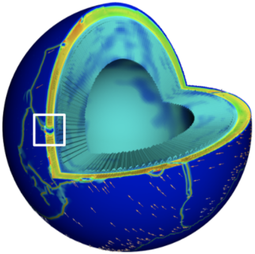
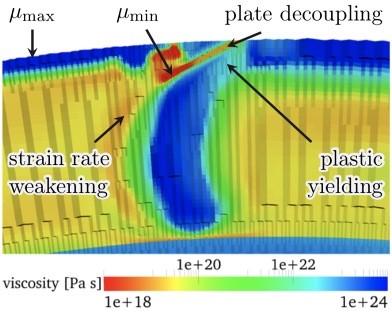
Plates are modeled as high-viscosity fluid (low T); plate boundaries are narrow zones of weak viscosity (prefactor $\omega(x)$ controls plate coupling).



Solver and discretization

- Discretization with adaptive finite hexahedral elements
- Order 2 or 3 for velocity, and corresponding stable discontinuous pressure elements of lower order
- Mesh refinement used to resolve narrow weak zones, and dynamically weakening areas (hinges)
- Resolved model has about 500M dofs, smallest mesh elements 1-2km (runs on several K CPUs)





Solver and discretization

- Nonlinearity is treated with Newton's method; plastic rheology (von Mises) uses Newton modification method to improve convergence

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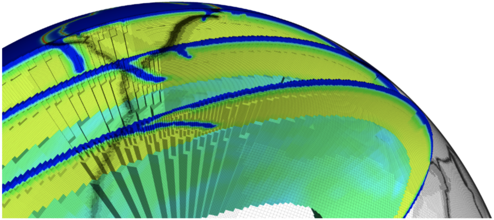
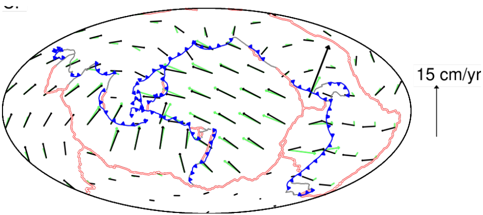
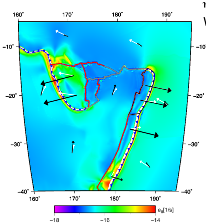
- Linearized Stokes solved with BFBT Schur complement, and geometric+algebraic multigrid-preconditioned GMRES

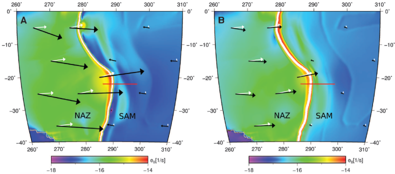


Setup and resolution

- Free-slip boundary conditions
- Input is temperature structure and fault weak zones (details later)
- Shown below: Misfit with plate observations (in green), trench rollback, plate coupling (white are observations)







From forward towards inverse modeling

- Hand-tuning models to fit observations cumbersome, only possible for a few parameters
- It is subjective
- Even if good parameters are found, we have no information about their stability or other possible parameters

Inverse problem theory (deterministic or Bayesian) is a systematic approach to infer parameters and to quantify our confidence in them.

Inverse Problems: Inference from data+model

Bayesian inversion for uncertain parameters $\boldsymbol{m}$

- Observations $\boldsymbol{d}$: plate motions; average viscosity below lithosphere,...
- Parameters $\boldsymbol{m}$: strain rate exponent n, yield stress τ_{y}, activation energy, prefactors in lower/upper mantle, weak zone factors w_i (about ~ 20)

$$\mu(T, \boldsymbol{u}) = \mu_{\min} + \min\left(\frac{\tau_{\text{yield}}}{2\dot{\varepsilon}(\boldsymbol{u})}, w \min\left(\mu_{\max}, a(T)\,\dot{\varepsilon}(\boldsymbol{u})^{\frac{1-n}{n}}\right)\right)$$

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- Model $f(\boldsymbol{m})$: Notation for nonlinear Stokes solve given $\boldsymbol{m}$, "parameter-to-observable map"
- Additive Gaussian data noise: $\boldsymbol{d} = f(\boldsymbol{m}) + \boldsymbol{e}, \boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, C_n)$.
- The posterior density is (using Bayes' Theorem):

$$\pi_{\text{post}}(\boldsymbol{m}) \propto \exp\Big(-\frac{1}{2}\|f(\boldsymbol{m}) - \boldsymbol{d}\|^2_{C_{\text{n}}^{-1}} - \frac{1}{2}\|\boldsymbol{m} - \boldsymbol{m}_0\|^2_{C_{\text{pr}}^{-1}}\Big).$$

Approximations of the posterior pdf $\pi_{\text{post}}(\cdot)$

With these assumptions (Gaussian prior and noise), the posterior is:

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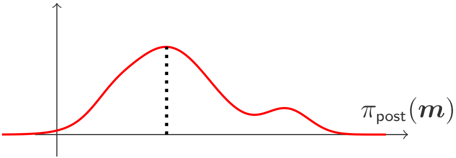
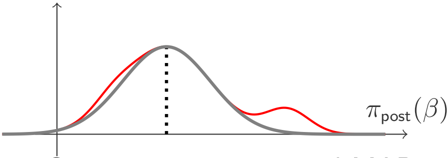
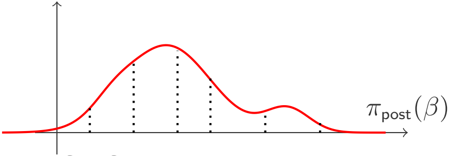


MAP estimation



Gaussian approx. around MAP



MCMC or importance sampling

In high dimension, we use...

- ...MAP estimation (i.e., Stokes-constrained optimization)
- ...Gaussian approximation of posterior

Inversion Algorithms

MAP optimization: Maximizing the posterior equals minimizing its negative log, i.e.:

$$\min_{\boldsymbol{m}} F(\boldsymbol{m}) := \frac{1}{2}\|f(\boldsymbol{m}) - \boldsymbol{d}\|^2_{C_{\mathsf{n}}^{-1}} + \frac{1}{2}\|\boldsymbol{m} - \boldsymbol{m}_0\|^2_{C_{\mathsf{pr}}^{-1}}$$

Compute gradient of $F(\cdot)$ using adjoint method!

Adjoint method to compute $F'(\boldsymbol{m})$:

1. For given $\boldsymbol{m}$, solve nonlinear Stokes equations:

$$-\nabla \cdot \left[\mu(T, \boldsymbol{u}, \boldsymbol{m}) \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top}\right)\right] + \nabla p = \boldsymbol{f}(T)$$
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2. Solve an adjoint problem for the linear (!) adjoint velocity $\boldsymbol{v}$

$$
\begin{aligned}
-\nabla \cdot \left[\mu'(T, \boldsymbol{u}, \boldsymbol{m}) \left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\top\right)\right] + \nabla q &= \partial F / \partial \boldsymbol{u} \\
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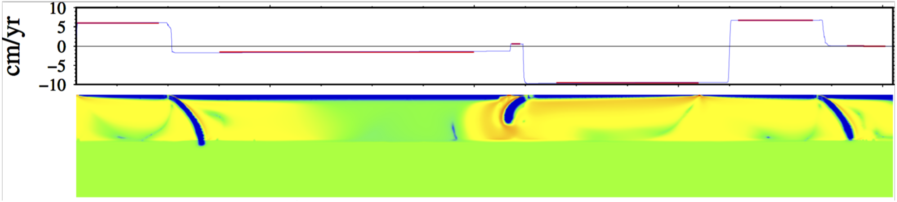
Requires one additional linear solve only!

Inversion Algorithm

- We also compute directional second derivatives of $F(\cdot)$ using similar approach
- Solve optimization (MAP point) using iterative descent method
- At MAP estimate, compute Gaussian approximation of posterior using that inverse Hessian approximates covariance matrix

Example I: Synthetic models



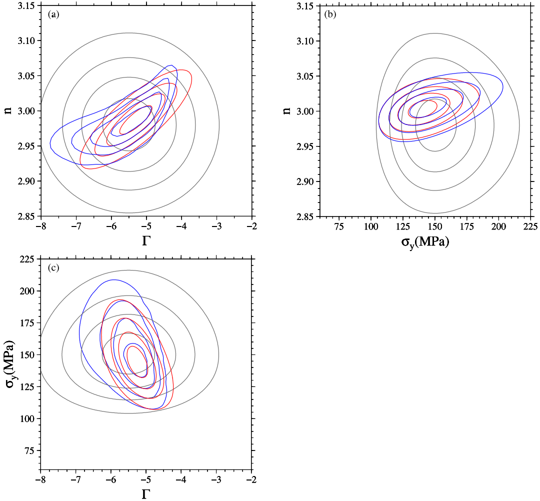
Parameters

- Parameters: plate coupling strength, global rheology parameters (≤ 5 params)

Observation data

- Data: plate velocities
- Computed from forward simulation (i.e., synthetic data)

Example I: Inversion of plate coupling & rheology params



2D pairwise conditionals for Γ, n, σ_y. Prior, Gaussian approximation and true posterior.

- Inference for plate coupling Γ (w) and rheology parameter n from surface velocity data.
- Gaussian prior distribution for parameters.
- Posterior approximation uses inverse Hessian as covariance matrix.
- Full posterior computed using Delayed Rejection Adaptive Metropolis (DRAM) sampling.

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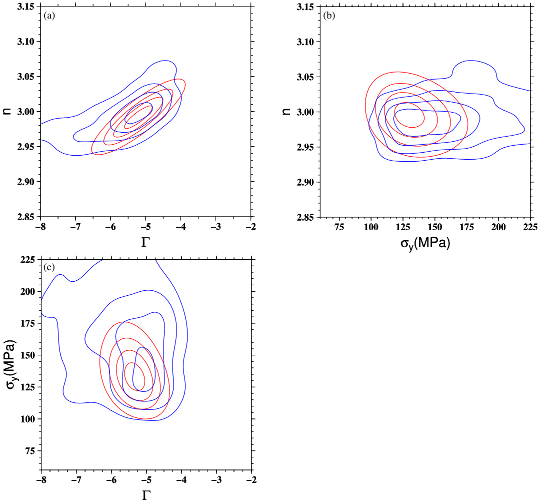

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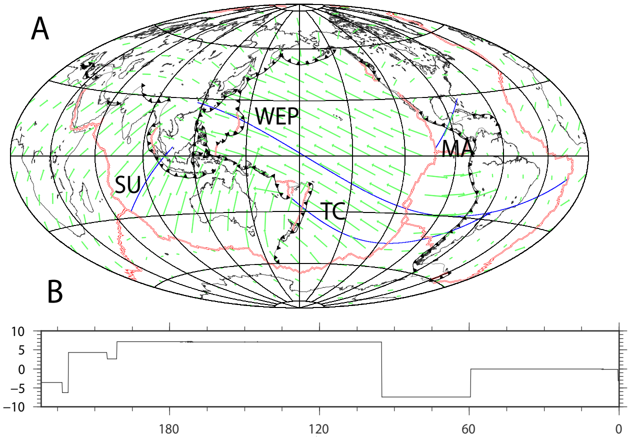
Updated input data and 2D inversions

- Mega-thrust interface using Slab1.0
- Thermal structure from seismic models and convergence rate of slabs
- Lithosphere model
- Slab structure blend with tomography-based models
- Morvel56



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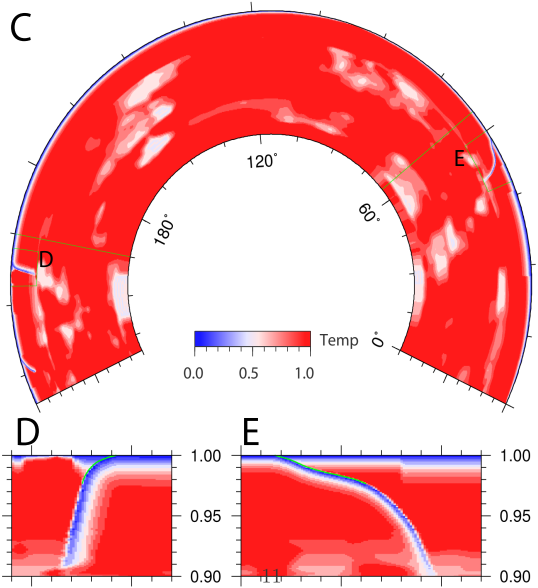
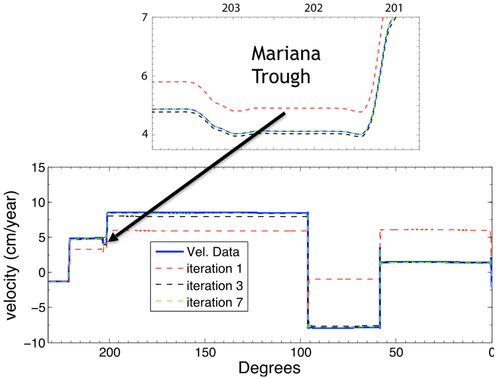


Example II: Inversion results for WEP slice

This is a 3D thin slice



- Fitting the observations after about 7 inverse problem iterations.

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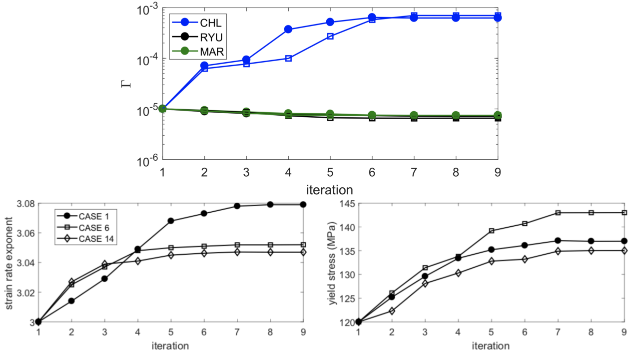

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- Convergence history for weak zone parameters (Chile, Ryukyu, Mariana), for strate rate exponent and yield stress.

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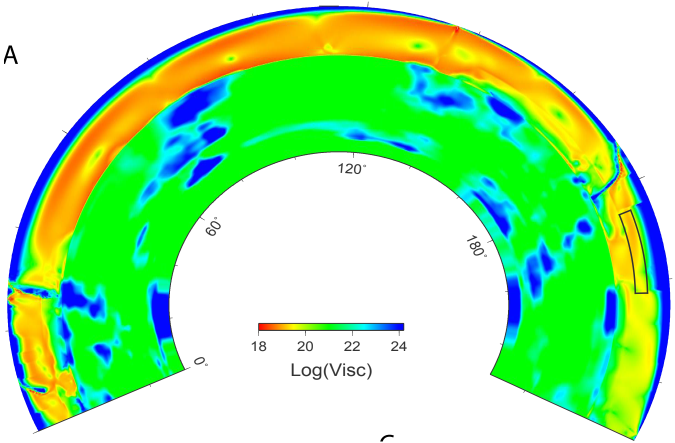

Effective viscosity. Here, besides plate motion data, we have also used the average viscosity in the boxed region.

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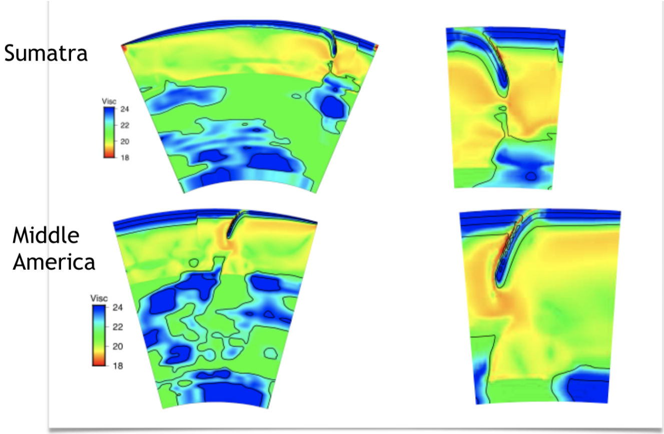

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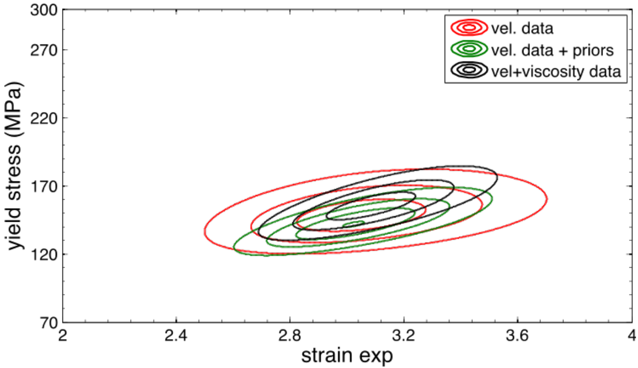

Two-dimensional distribution between shear stress and n with and without using average viscosity data.

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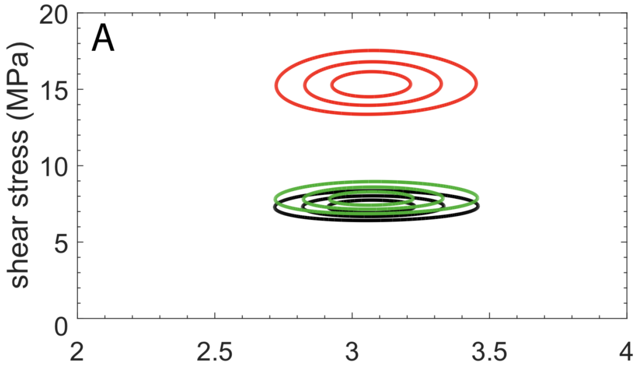

Shear stress estimates (with uncertainty) for Chile (red), Ryukyu (black) and Marianas (green), versus strain rate coefficient n.

Summary & Perspectives

Solvers

- High-resolution adaptive discretization
- Full Newton solver for instantaneous Stokes with strain-rate weakening and yielding
- Consistent temperature field (slabs) and faults; Self-consistent flow field

Inversion

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- Gives confidence/uncertainty regions, and trade-offs
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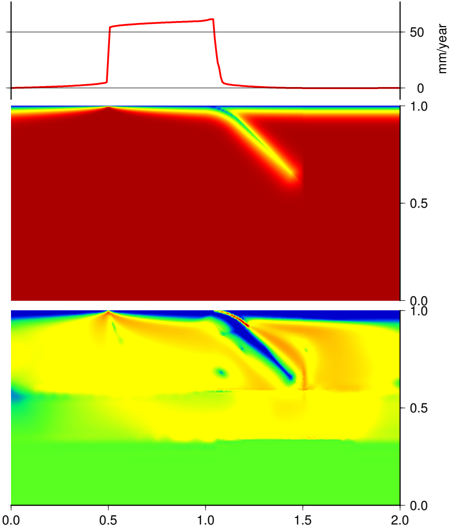
Thanks!

Extra Slides

Inversion of initial temperature T_0 (and global parameters)

Forward simulation



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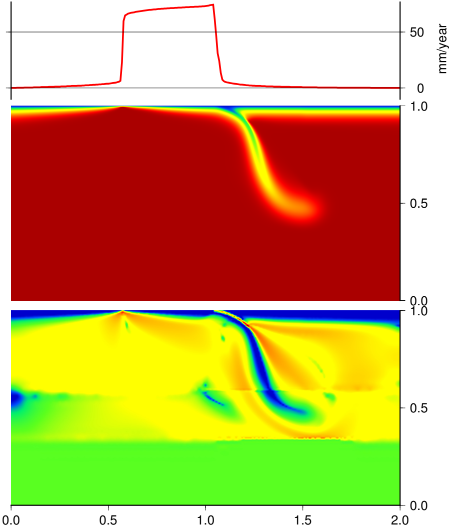

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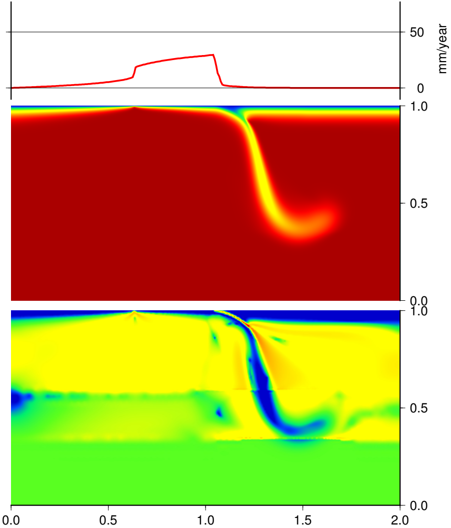

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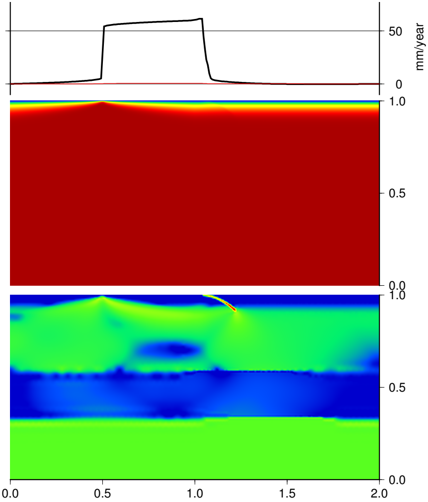

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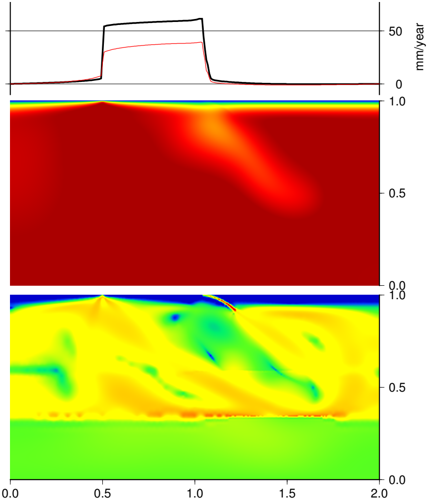

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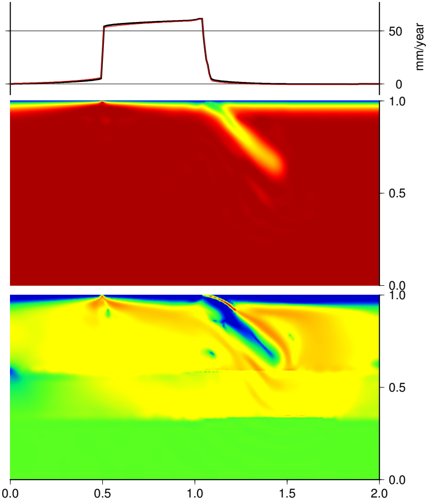

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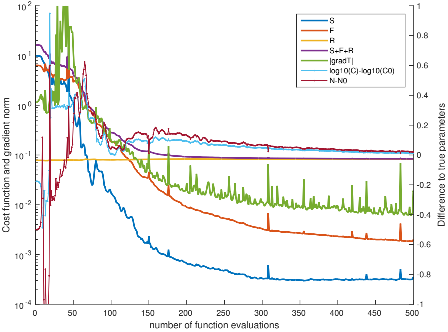

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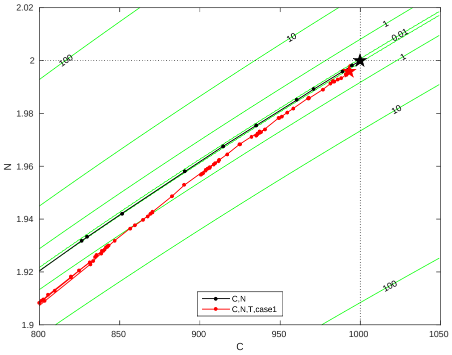

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- I'd like convergence to be faster
- Part of reason for slow convergence is flat valley in cost landscape indicating trade-off between parameters